Imperial College London

Can the evolution of financial markets be explained with fluid mechanics?

ME3 Literature Research Project

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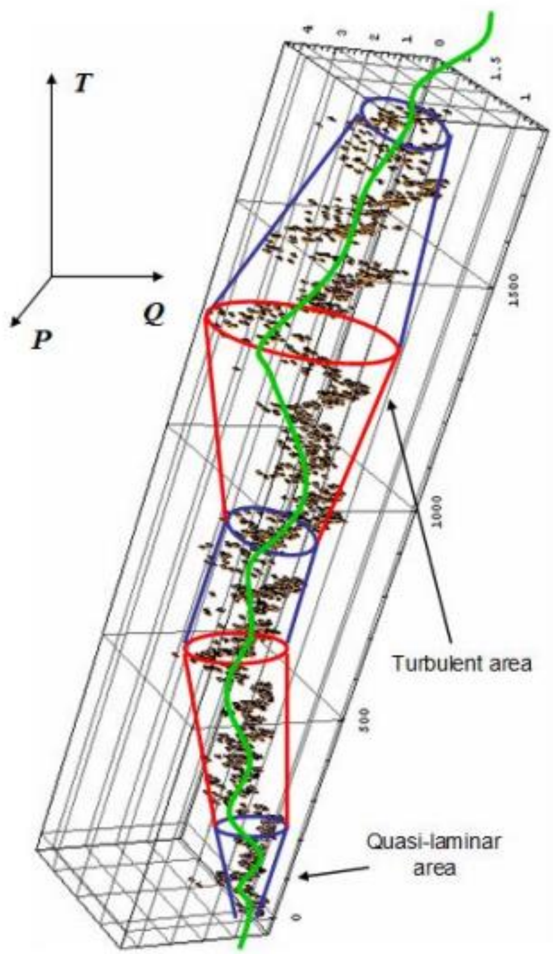

Three-dimensional rotational trajectory of WIG index quoted on the Warsaw Stock Exchange, with truncated rotary quadric surfaces moving in precession in the period from 1994-04-18 to 2002-04-18 drawn in space $R3^{+} = P \times Q \times T$ (index value, volume, and time, respectively) (48).

Abstract

It is well acknowledged that chaos is found in both the financial market and fluid flow. Some of their most important relationships as well as the rules controlling financial markets are covered in this literature review. An overview of the Black-Scholes model and the basic mechanism of the corresponding option pricing was given. Along with CAPM, two non-linear speculative autoregressive models, the ARMA and ARCH group, were also introduced. Current econometric models are developed using particle motions such as the Geometric Brownian Motion, and the Black-Scholes PDE is a slight variant of the heat equation. The stock market could be studied similarly to a fully developed turbulent flow by taking into account intermittency, which are high bursts of volatility clusters, and nonextensivity, which is the scaling effect seen in eddies. The inadequacy of conventional algorithms to incorporate stochastic fractals and inhomogeneity is regarded as the major obstacle in minimising prediction error. Examples of outstanding fluid mechanics approaches in modelling and forecasting financial evolutions from previous literature were highlighted and the economic analogue of Reynolds number and viscosity are considered to be alternative measures to dispersion and uncertainty. Unarguably, compared to the application of fluid theory to finance, more investments have been drawn to the development of pure statistical approaches over the past decade. Nevertheless, it should not be neglected that fluid characteristics could offer additional insights into problems that remain unsolved in traditional economic models.

Keywords: Financial market; fluid mechanics; Black-Scholes; CAPM; autoregression; ARMA; ARCH; turbulence; intermittency; nonextensivity; stochastic fractals; Brownian motion

Statement of objectives

The objectives of this literature review are defined as follows:

- *provide an overview of the laws of financial markets and the corresponding mathematical formulation.*
- *establish whether a link with the laws of fluid mechanics is possible or not*
- *discuss the chaotic nature of the financial markets and build a link with fluid turbulence.*

Imperial College
London

DEPARTMENT OF
MECHANICAL ENGINEERING

2022-2023 ME3 LITERATURE RESEARCH PROJECT

Meeting Log

STUDENT:	CHU LAP KIN ISAAC	**SUPERVISOR:**	ANDREA GIUSTI
TITLE:	Can the laws of ~~fluid mechanics~~ financial market be explained with fluid mechanics?		

Meeting	Date	Student Signature	Supervisor Signature
Week 2	14/10/22	[illegible]	[illegible]
Week 5	01/11/22	[illegible]	[illegible]
Week 7	17/11/22	[illegible]	[illegible]
Week 9	01/12/22	[illegible]	[illegible]

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1 Introduction

It is widely recognised that the financial market does not behave in a predictable manner. Mathematicians have been attempting to study the chaotic behaviour of the financial market by deriving quantitative models. The earliest application of physics and mathematical models could be dated back to 1900 when Louis Bachelier's PhD thesis provided a model to price options based on random walk and assumed that the price variations follow a normal distribution (1). In the 1960s, Edward Thorpe, a New Mexico State University mathematics professor, used probability theory from his research and created a statistical system to win Blackjack games (2). In the 1960s stochastic calculus was introduced to the study of finance by Paul Samuelson, whereas Merton implemented it to a continuous time process (*Weiner Process*) (3). This led to the development of the *Black-Scholes model*, which allow us to compute the fair price for a *European option* and was awarded the 1997 Nobel Memorial Prize in Economic Sciences (4).

However, the econometric models mentioned above heavily rely on assumptions and neglect some critical characteristics for price deviations, such as the presence of crashes, nonlinear serial dependence, etc. Numerical simulations have shown evidence that the stock market behaves differently than what is described in the existing models (5, 6). Econophysics is an interdisciplinary research field where theories in physics and fluids are applied to solve problems in economics. It was introduced in the mid-1990s due to the fact that physicists were not satisfied with the traditional explanations of the financial market, which had been proved to have inaccurate results empirically. Numerous studies have confirmed and exploited similarities between financial market behaviours and physical phenomena. For instance, the study of *stochastic fractals* from fluid turbulence was used to model *intermittency* in interest rates, the *Kolmogorov Cascade* was applied to the foreign exchange (FX) market to explain the heterogeneities between agents and their consequences on flow phenomenon at varying time scales, and the Black-Scholes equation was transformed into an analogue of the heat equation (7, 8).

This paper is organised as follows. Section 2 introduces several fundamental governing equations in financial markets and relevant background information. Section 3 focuses on explaining how the relationships between the financial market and physics, more specifically fluid mechanics, are developed. Sections 4 and 5 are the discussion and conclusion respectively.

2 Current financial models

2.1 Black-Scholes model

2.1.1 Background

The Black-Scholes Model could be considered the most important equation governing the market dynamics in the world of finance. The model is based on a parabolic partial differential equation, the *Black-Scholes equation*, where the *Black-Scholes formula* could be deduced and is able to give the theoretical estimate price of European-style options. This equation is named after economists Fischer Black and Myron Scholes, where the original idea from Robert C. Merton was further developed and sometimes credited (9, 10). The key idea behind the model is that options could be used and the risk in a portfolio could be hedged by buying and selling the underlying assets in a specific way. The method has led to the development of *continuously revised delta hedging*, which acts as a fundamental of other complex hedging strategies adopted by investment banks and hedge funds (11).

2.1.2 Options

An *option* is a financial instrument that gives the right, but not the obligation, to buy or sell an asset, within a certain period of time and subject to certain conditions. Options are often called *derivatives* since their prices are derived based on the value of the underlying securities such as stocks (9). The majority of options in the market can be classified into the following: *American Options* and *European Options*. An American option can be exercised at any time up to the date of expiry, whereas a European option can only be exercised on the agreed expiry date. Any outstanding options must be exercised before the expiration date, it is sometimes called the *maturity date.* The *striking price* or the *exercise price* refers to the price that is paid when the option is exercised. The simplest kind of option is *call options*, which gives the option holder the right to buy a single share of common stock (10). Whereas *put options* give the holder the right to sell a single share of common stock. For simplicity, we will be focusing on call options throughout this section.

In general, if the price of the underlying stock is higher than the exercise price, the option would almost surely be exercised, and a profit could be taken as the difference between the stock price and the exercise price (9). On the other hand, if the stock price falls under the exercise price, the option can only be left until it expires without being exercised. Under normal circumstances, the relationship between a stock and its underlying option is shown in figure 1. The value of the stock option for successively shorter maturities is shown through the lines T1, T2, and T3. The value of the option could not be higher than the stock, hence the maximum price of the option is represented by line A, 45 degrees from the axis (9). The minimum price of the option is represented by line B, and it cannot be lower than zero and cannot be smaller than the stock price minus the exercise price. It can be concluded that options have a higher volatility than stocks. In other words, a percentage change in the stock price would result in a larger percentage change in the option (9).

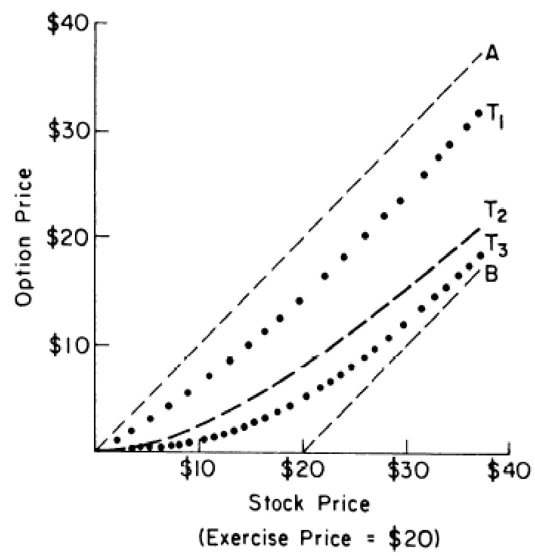

Figure 1. The relation between option value and stock price (9)

2.1.3 Evolution of option valuation

There were numerous attempts before Black and Scholes to work on a valuation model for options and they have been expressed in terms of warrants, which refer to options that allow the holder to buy shares in equity rather than shares of common stock (9 ,12). However, arbitrary parameters are involved in these attempts and hence are incomplete. For instance, Sprenkle's formula for option pricing can be expressed by the following (9):

$$kxN(b_1) - k^*cN(b_2) \quad \text{Eq.1}$$

$$b_1 = \frac{\ln\frac{kx}{c} + \frac{1}{2}v^2(t^* - t)}{v\sqrt{(t^* - t)}} \quad , \quad b_2 = \frac{\ln\frac{kx}{c} - \frac{1}{2}v^2(t^* - t)}{v\sqrt{(t^* - t)}} \qquad \text{Eq.2}$$

Where x, c, t^*, t and v^2 represents the stock price, exercise price, maturity date, current date and the variance rate of the return on stock respectively. ln is the natural logarithm and $N(b)$ is the cumulative normal density function. Sprenkle attempted to estimate empirically the unknown parameters k and k^* but were unable to do so. Similar issue was faced by Samuel and Merton, where they were trying to deduce constants in order to estimate the discount rate of a warrant (12). Ultimately the Black-Scholes model was built upon the empirical valuation formula derived by Thorp and Kassouf (13). The detail derivation could be found in the Nobel Prize-winning paper *The Pricing of Options and Corporate Liabilities* (9). The final result of the Black-Scholes formula is shown below:

$$C = SN(d_1) - Ke^{-rt}N(d_2) \qquad \text{Eq.3}$$

$$d_1 = \frac{ln\frac{S_t}{k} + \left(r + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} \quad , \quad d_2 = d_1 - \sigma\sqrt{t} \qquad \text{Eq.4}$$

Where C, N, S_t, K, r, t and σ is the call option price, cumulative distribution function of the normal distribution, spot price, strike price, risk-free interest rate, time to maturity and the volatility of the underlying asset respectively.

2.2 Speculative models

2.2.1 Background

In recent decades mathematicians, economists and investors have been competing to correctly determine the future value of company stock or other financial instruments in order to maximise return. Some speculative strategies include value investing, where investors would purchase an asset at a discount on its intrinsic value, time series models, which use historic data as a driving metric for financial forecasting, econometric models, which utilise complex data and mathematics throughout the process and the list goes on (14). In fact, some of these forecasting models, such as the Capital Asset Pricing Model, serve as an alternative derivation for the Black-Scholes model mentioned in section 2.1 (9). The Capital Asset Pricing Model (CAPM), Autoregressive Conditional Heteroskedasticity (ARCH), and Autoregressive Moving Average (ARMA) models are introduced in this section to provide a better understanding for the context of this review as they are considered the common methods to validate or compare the relationship between fluid mechanics and the financial market.

2.2.2 Capital Asset Pricing Model (CAPM)

Building on Harry Markowitz's earlier research on diversification and modern portfolio theory, Jack Treynor, William F. Sharpe, John Lintner, and Jan Mossi each independently established the CAPM (15, 16). For their contribution to the field of financial economics, Sharpe, Markowitz, and Merton Miller shared the 1990 Nobel Memorial Prize in Economics. Another variation of the CAPM, known as Black CAPM or zero-beta CAPM, was created by Fischer Black in 1972 and does not rely on the existence of a riskless asset (9). This variation, which shows more consistency with empirical verification, contributed to the CAPM's wide adoption and could be expressed in the following form:

$$E(R_i) = R_f + \beta_i(E(R_m) - R_f) \quad \text{Eq.5}$$

Where $E(R_i)$ is the expected return on the capital asset, R_f is the risk-free rate of interest, β_i is the sensitivity of the expected excess asset returns to the expected excess market returns, and $E(R_m)$ is the expected return of the market (e.g., S&P 500). $E(R_m) - R_f$ is the difference between the expected market rate of return (market premium). CAPM is derived under assumptions such as zero transaction costs and all assets being perfectly divisible and liquid (17). The result demonstrates that the cost of equity capital is solely determined by beta. Despite the CAPM's numerous failures in empirical testing, mainly caused by beta bias, and the existence of more recent methods for valuing assets and portfolio selection such as arbitrage pricing theory and Merton's portfolio problem, it is still widely used because of its ease of use and versatility (18). A visualisation of the result of CAPM could be reviewed in figure 2. The y-axis shows the expected return, while the portfolio risk, is represented by their standard deviations on the x-axis. The return of a riskless asset and risky assets are represented by their *efficient frontiers*, which are deduced by plotting returns of the underlying assets against their risks, can be seen as the line from R_f to T and the curve abc respectively. The line constructed from R_f to g shows an asset allocation with both riskless asset and risky assets (i.e. stocks and bonds).

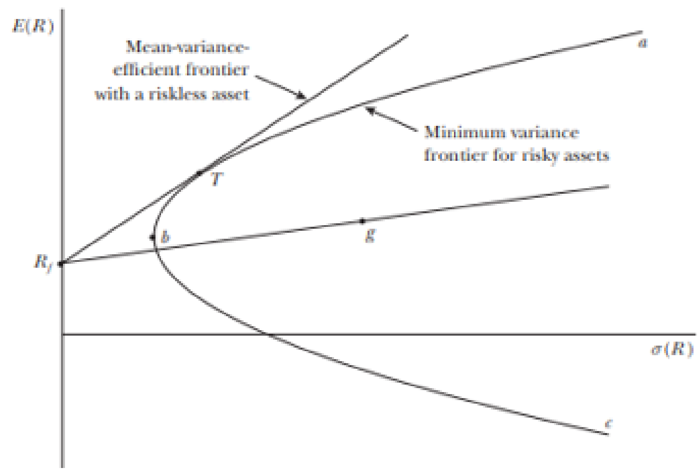

Figure 2. Efficient Frontiers of different portfolios (18)

2.2.3 Autoregressive Moving Average (ARMA) models

ARMA models are often used in statistical time series analysis and provide a concise description of stationary stochastic processes. Stochastic processes are frequently observed in the field of statistics and finance, which refers to the property of being well described by a random probability distribution (19) . The ARMA model consists of two polynomial terms, which are the *autoregression* (AR) and *moving average* (MA) terms respectively. The ARMA model serves as a technique for comprehending and forecasting future values in a time series of data. The AR component refers to the process of regressing the variable on its own lagged, or in other words, prior values. Whereas the MA component accounts for the error which is modelled as a linear combination of error terms deduce from previous averages calculated in a shifted time window (19, 20). Based on the Laurent series and Fourier analysis, the ARMA model was derived in Peter Whittle's thesis in 1951 and was popularised by George E.P Box and Jenkins, who created an iterative (Box-Jenkins) method for selecting and estimating the model (21). This method was proved to be more accurate for low-order polynomials and could be expressed by the following notation:

$$X_t = \varepsilon_t + \sum_{i=1}^{p} \varphi_i X_{t-i} + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i} \qquad \text{Eq.6}$$

Where $\sum_{i=1}^{p} \varphi_i X_{t-i}$ and $\sum_{i=1}^{q} \theta_i \varepsilon_{t-i}$ represents the AR and MA terms respectively. The autoregressive coefficient is denoted by φ_i, the moving average coefficient is expressed by θ_i, and ε_t is the error term. The order of ARMA models are often expressed in the form of ARMA(p,q) (20) . For instance, ARMA(1,2) is referring to an AR order of 1 and MA order of 2, which means the current autoregressive component is dependent on the previous value whereas the moving average component is dependent on the previous two values. The value of p and q could be simply adopted using relevant historical data or derived using methods that model the correlation between consecutive values using previous separations, including the autocorrelation function (ACF) or partial autocorrelation function (PACF) (20).

Under most circumstances, a generalisation of the ARMA model, the autoregressive integrated moving average (ARIMA), is adopted in time series analysis such as forecasting (20, 22). The ARIMA model is especially useful when the underlying data show signs of non-stationarity (i.e., data that consists of trends or cyclical behaviours). The "integrated" element within the ARIMA model corresponds to a differencing step, which may be applied once or more times to remove non-stationarity (20). The purpose of this is to tackle the inconsistency in predicting non-stationary time series, such as financial markets which exhibit strong seasonality, by turning it into a stationary process. The ARIMA model has a very similar form with the ARMA model and could be expressed by the following:

$$Z_t = \varepsilon_t + \sum_{i=1}^{p} \varphi_i Z_{t-i} + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i} \quad \text{Eq.7}$$

Notice that the X_t term in ARMA model is now replaced with Z_t, which is the difference between consecutive timepoints of the data series. The order of ARIMA models could be expressed as ARIMA(p,d,q), with d being the order of differencing, similar to what we have seen from finite difference methods (19). The application of these autoregressive models expands beyond the financial market, they are widely adopted in other time series analysis such as the forecasting of electricity load in a circuit or tourism demand of city (23, 24).

2.2.4 Autoregressive Conditional Heteroscedasticity (ARCH) models

Conditional variances can be modelled and predicted using Autoregressive Conditional Heteroskedasticity (ARCH) models (25). Similar to the ARMA models, ARCH models can be broken down into two components. “AR” stands for *autoregression*, and it was defined in section 2.2.3. “H” refers to *Heteroskedasticity*, which refers to the study of deviations, and “C” represents *Conditional*, in which the probability distributions of volatility change depending on the present value (25, 26). When it comes to error terms, ARIMA models deal with serial correlation, whereas ARCH models deal with the fact that the variance of prediction errors is not constant but varies over time.

The first few papers that looked at the statistical characteristics of stock returns were those by Mandelbrot and Fama (27). Engle worked on enhancing time-series analysis in the 1980s (25). Although volatile variables, like stock prices, can move dramatically over a period of time, most statistical approaches at the time viewed them as constants. Engle invented the statistical method known as ARCH, which leverages previously observed patterns of variance to forecast future volatility, after observing the variance of stock returns (25). The identification of Conditional Heteroskedasticity from Engle was recognised, and he was awarded a Nobel Prize in Economics in 2003. The prices and risks associated with investing in stocks are now calculated using improved ARCH models in banking and finance.

To better understand homoscedastic and heteroscedastic processes, the following scenario is set: let there be a univariate stochastic process Y. If the standard deviations of Y remain constant for all periods t, Y is said to be homoscedastic. It is considered heteroscedastic otherwise. If the unconditional standard deviations t are not constant, the process is unconditionally heteroscedastic. If the conditional standard deviations $\sigma_{t|t-1}$ are not constant, the distribution is conditionally heteroscedastic (25). The idea of heteroscedasticity could be illustrated using the following examples of stock market returns and the cost of electricity (26). Returns on bonds or stocks typically exhibit conditional heteroscedasticity. Although the volatilities of these prices are not consistent, periods of low or high volatility are typically unpredictable. On the other hand, the cost of electricity in New Delhi displays unconditional heteroscedasticity (26). Compared to other seasons, summer tends to have higher price volatility. Since this is predictable and the prices of electricity show unconditional heteroscedasticity.

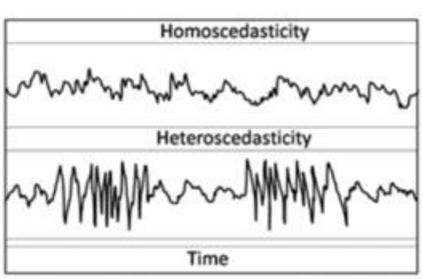

Figure 3: Homoscedastic vs. Heteroscedastic (26)

In terms of the expression of ARCH, it is a simple deviation from the AR model, the only difference is that the predicted variance residual, $\hat{\varepsilon}_t^2$, is now computed using the following (28):

$$\hat{\varepsilon}_t^2 = \hat{\alpha}_0 + \sum_{i=1}^{q} \hat{\alpha}_i \, \hat{\varepsilon}_{t-i}^2 \qquad \text{Eq.8}$$

In the field of statistics, the $\frown$ notation is often used to describe predictive values. $\hat{\alpha}_i$ is the residual coefficient and q could be derived from PACF and the order of ARCH lag is expressed in the form ARCH(q). There exist multiple improvements of ARCH models, such as the generalised ARCH (GARCH), Integrated GARCH (IGARCH), and Threshold GARCH (TGARCH) (26, 29). The most suitable models need to be determined on a case-by-case basis, for instance, the IGARCH and TGARCH were proven to yield the best performance in forecasting exchange rates (26).

3 Fluids in finance

3.1 Black-Scholes, conduction and diffusion

It is now possible to assess the correlation between the Black-Scholes equation and the heat equation given what is known about both equations. Numerous similarities exist between the Black-Scholes equation and the well-known heat equation (30, 31, 32). This similarity is the reason why there is such a profound correspondence. By investigating the Black-Scholes equation as the heat equation and ultimately arriving at a solution, a perspective could be obtained. Let’s recall the Black-Scholes PDE and the heat equation (31) :

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \qquad \text{Eq.9}$$

$$\frac{\partial F}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2} \qquad \text{Eq.10}$$

Given S as the value of the underlying asset, t representing time, V is the price of the option, r is the risk-free rate, the temperature F and position x. For the context of this literature review, the diffusivity term is represented by $\frac{1}{2}\sigma^2$ instead of using the thermal conductivity, density and specific heat capacity of the material. τ is defined as difference between the terminal value of time, T, and the present time value. Several popular methods exist for transforming one equation to the other, which includes applying assumptions using the finance rationales and executing dimensional analysis on the variables (31, 32). In this section we will include a brief introduction to the former as it gives a more relatable intuition with the heat equation.

The transformation is based on an important step named *time reversal* (31). The purpose of this is to change the backward operator in the Black-Scholes equation into a forward operator, which could be defined through the signs within the equation. This is also consistent with the heat equation's forward operating nature. By doing time reversal would yield the following substitutions (31):

$$x = ln\tilde{S}_\tau + (r - \frac{1}{2}\sigma^2)\tau \quad \text{Eq.11}$$

$$F_\tau = \tilde{V}_\tau e^{r\tau} \quad \text{Eq.12}$$

And the boundary condition of option price:

$$V_T = \max(S_T - K, 0) \quad \text{Eq.13}$$

In this context, τ expresses the time until maturity such that the option expires when $\tau = 0$. In this time reversed domain, $\tilde{S}_\tau$ and $\tilde{V}_\tau$ represents the current stock price and option price respectively (31). S_T denotes the stock price at maturity using the known solution of Geometric Brownian motion and Eq. 13 is derived using the nature of option contracts as explained in section 2.1.1 (9, 33). The above substitutions, boundary and terminal conditions are the key steps in converting between the Black-Scholes equation and heat equation. For simplicity, the remaining algebraic manipulation is not written in detail in this review.

A previous literature by Leonard Mushunje analysed the stock market using the heat equation in a different way. He suggested that mathematicians tend to focus on the volatility of stock prices, whereas the *diffusivity* factor has always been neglected. By considering stock prices as *martingales* and as *Markovian,* a model for expected stock prices was derived (34). The *drift* and volatility components of stock price and heat diffusion are compared, and a decent explanation of the diffusivity of stock prices could be found (34). In another paper *Semi-closed form solutions for barrier and American options written on a time-dependent Ornstein Uhlenbeck* (OU) *process,* the form of heat equation is utilised to derive a pricing model for *barrier options,* adapting the use of heat potentials that was adapted by A. Lipton in the 20th century (35, 36). The OU process is a widely used stochastic process in financial mathematics and physics and its original application includes modelling for the velocity of a massive Brownian particle under the influence of friction (37). Typical uses of such PDE with moving boundaries could be found in nuclear power engineering combustion, solid-propellant rocket engines, phase transitions, crystal growth etc. (35).

3.2 Multifractals, intermittency, and turbulence

It is widely acknowledged that the financial market series shows non-linearity (38), and recent studies have shown that this non-linear fluctuation in foreign exchange markets exhibits fractal features (39). This statistically translates to non-integer dimensionality, in other words, intermittency or inhomogeneity in a time series. The term *fractal* was previously mostly used to refer to deterministic chaos, which is created by a small number of generating equations (40). However, a new class of stochastic fractals was discovered beginning in the 1980s, mostly as a result of physics research on turbulence. In high-dimensional systems, there exist processes known as *multifractals* (40, 41). The idea of stochastic fractality is directly applicable in econometrics, despite the fact that the mechanisms that produce multifractals in physics may not always correspond to economic processes (40). Multifractals in physics possess a very strong scaling symmetry, whereas fractals found in financial markets demonstrate weak scaling symmetries and evolve toward a non-fractal state over time (41). They hold much shorter intervals and are called stochastic fractals.

In *The fractal structure of exchange rates: measurement and forecasting,* Gordon *et al.* suggested that fractal properties could be found in the determinants of exchange or interest rates and differential in real rates of return (8). Based on state transitions, a forecasting algorithm is derived and backtested against various time series models, including those introduced in section 2.2. It is useful here to understand the fundamental parameters of fractals $H, \alpha,$ and C, which corresponds to nonstationary, the probability distribution, and the degree of intermittency (42). H typically ranges from 0 to 1, where the underlying non-linear series is considered as long-term memory series when its value is smaller than 0.5, and antipersistent or turbulent processes otherwise. α ranges from 0 to 2, with $\alpha = 2$ and $\alpha = 1$ representing a lognormal and Cauchy distributions respectively. In simple words, a smaller α would lead to a smoother process (8, 27). C represents *codimension,* the series is considered to be homogenous when $C = 0$ and inhomogeneous or fractal otherwise. These three parameters could be computed for non-linear series to analyse its fractality through a scaling procedure, which is by all means taking rations of stochastic processes (27, 41, 43):

$$\mu(|ln\, Y_t - \ln Y_{t-1}|^q) \approx \mu(|ln\, Y_t - \ln Y_{t-T}|^q)[\left(\frac{t}{T}\right)^{\zeta(q)}] \qquad \text{Eq.14}$$

Where q is a series of exponents, Y_t is the time series itself, μ represents the mean and T is the terminal value of time. The exponent ζ arises as a product of scaling and is a function of q and has the following form (8):

$$\zeta(q) = qH - \left\{\left[\frac{C}{\alpha-1}\right](q^\alpha - q)\right\} \text{ for } \alpha \neq 1 \qquad \text{Eq.15}$$

$$qH - [Cqlnq] \text{ for } \alpha = 1$$

The degree of curvature for the slope of $\zeta(q)$ is a function of probability distribution and inhomogeneity, and hence the measure of the turbulence of the process (8). Using the above operations, the ARIMA, GARCH and state-transition ARCH (ST-ARCH) showed fractality in 70% of the simulation tests. 7500 data points of interest rates and exchange rates series were also tested and they all behave strongly as fractals, with values of C topping 0.145. These scaling symmetries multifractal processes could be exploited, and the following forecasting model is derived (8):

$$lnY_t = a_0 + a_1 lnY_{t-1} + a_2\vartheta_{2t-1} + a_3 S_{Et} lnY_{t-1} + a_4 S_{Et}\vartheta_{2t-1} \qquad \text{Eq.16}$$

Where S_{Et} denotes state variable representing extreme fluctuations, a_i is the regression coefficient and ϑ_i is the proportionality coefficients obtained by dividing the log difference of previous data points in the time series. This state transition approach was backtested against other econometric models on the FX market. The models are estimated using the first 500 data points and forecasting methods are applied for the remainder (8). It is striking to observe that the state transition model tires or have an advantage over the AR(1) or GARCH models two-third of the time.

The intermittency, volatility clusters such as energy bursts and volatile movements in finance, and nonextensivity, the anomalous scaling of properties like entropy, are discussed using a slightly different approach in the paper *Intermittency and Nonextensivity in Turbulence and Financial Markets* and suggested that a single parameter q from *nonextensive thermostatistics* could relate all the above properties (44, 45). Recall the study of turbulent flows is characterised by the statistics of velocity differences $v_r(x) = v(x) - v(x + r)$ at different scales r. The scaling invariance become trivial when examining the cascade process of kinetic energy dissipation through smaller and smaller hierarchy of eddies in to heat, while the energy cascade is governed by the underlying PDF of v_r.The PDFs are normally distributed under large scales $(\sim L)$, but exhibit strong non-Gaussian and fatter wings than expected for values far from *integral scale,* which is a strong sign for intermittency phenomenon. This so called *PDM problem* attracted several attempts (39, 45), and we will review the simplest and most accurate solution proposed by Tsallis, who gave a generalization of *Boltzmann-Gibbs thermostatistics* based on the scaling properties of multifractals (44):

$$p_q(x) = [1 - \beta(1-q)x^2]^{1/(1-q)}/Zq \qquad \text{Eq.17}$$

Where the normalisation factor for $1 < q < 3$ could be expressed by:

$$Z_q \equiv \left[\frac{\beta(q-1)}{\pi}\right]^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{\mathrm{q}-1}\right)}{\Gamma(\frac{3-\mathrm{q}}{2(\mathrm{q}-1)})}$$

This model is checked with the turbulence statistic data (45), and the results agree with all spatial scales and normalised velocity differences in all orders. The same approach could be adopted directly to the financial market, where we focus on differences on prices instead of velocities. The model provides matching behaviour with the statistics of price differences over all temporal scales, see figure 5 below. However, its integral value of 2.2 days is not consistent with other estimates produced using similar approach in other studies (39).

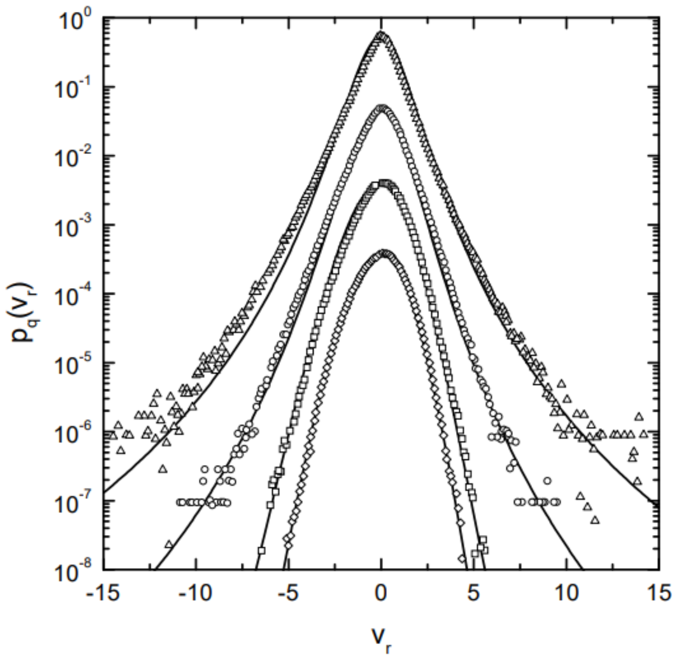

Figure 4a. Data points: standardized probability distribution pq(vr) of velocity differences vr(x) = v(x) − v(x + r) for spatial scales r = 0.0073L, 0.0407L, 0.3036L, 0.7150L, with L/η = 454 and L and η being, respectively, the integral and Kolmogorov scales; data taken from (38), provided by Chabaud et al.13; Solid lines: least-squares fits of modified PDF (1); from top to bottom: q = 1.26, 1.20, 1.11, 1.08; β^- = 0.69, 0.66, 0.55, 0.62; β^+ = 0.88, 0.82, 0.76, 0.70 (for better visibility the curves have been vertically shifted with respect to each other) (44).

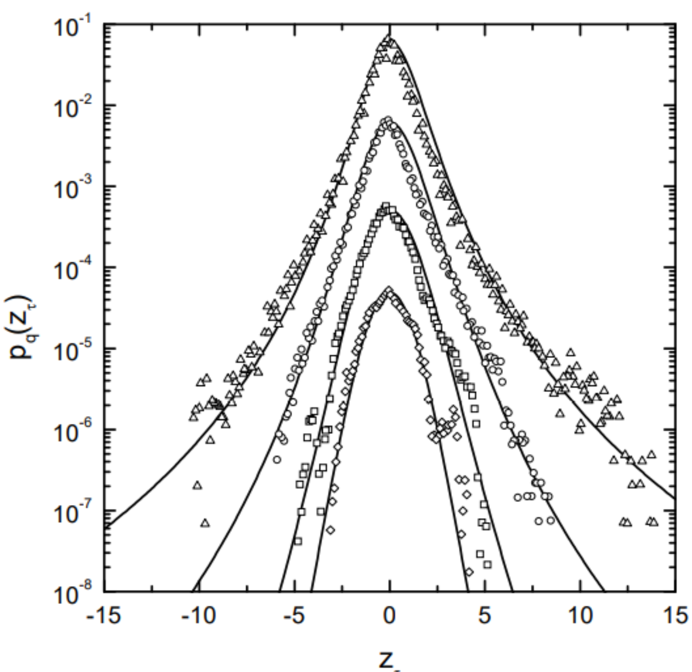

Figure 4b. Data points: standardized probability distribution pq(zτ) of price differences zτ = z(t) − z(t + τ) for temporal scales τ = 0.0035τL, 0.0276τL, 0.2210τL, 0.8838τL, with τL = 186265 s being the integral scale; data taken from (38), provided by Olsen & Associates; Solid lines: least-squares fits of modified PDF (1); from top to bottom: q = 1.35, 1.26, 1.16, 1.11; β^- = 1.12, 0.83, 0.75, 0.75; β^+ = 0.98, 0.72, 0.61, 0.77.(for better visibility the curves have been vertically shifted with respect to each other) (44).

3.3 Financial agents and soft matter

Apart from the fluid turbulence perspective as discussed in the previous section, one can also relate the financial market with fluid mechanics by viewing the stock market as a physical system analogous to a fluid that is evolving in a macroscopic space, and subject to a force that affects its movement over time arises from the collision of the supply and demand of financial agents. In the field of fluid mechanics, these forces are typically analysed using physical properties such as density, viscosity and surface tension (46, 47).

In a recent literature *Stock market's physical properties description based on Stokes' law* by Geoffrey Ducournau (47), the dynamics of the stock market behaviour is explained qualitatively and quantitatively from the supply and demand collision, which results from the physical characteristics of financial agents as outlined by the *Stokes Law*. As opposed to analysing the fractality of time series from last section, whether the "flow" of the stock market is laminar, turbulence, or transitionary could be determined by reconstructing the econophysics analogue of the Reynold number and viscosity (13). To derive these values, the stock market system could be categorised into two elements, which are stock market obstacles and stock market fluids, while both consist of financial agents with varying properties. Financial agent could be defined as a particle which have three coordinates *Side*, *Price*, and *Size* (47). Whereas stock market obstacles are defined as pending orders with a predetermined buying or selling price such as a *Limit Order* or *Stop Order.* The stock market fluid itself is modelled as a particle with changing properties with respect to time, which is equivalent to *Market Order*, that takes the current market price as the price of the particle (47). The interaction between the fluid and obstacle lead to the flow in stock market, whereas these interactions could be distinguished into two types: *Active Interaction* takes place when the order-book's Bid/Ask price collides with fluids particles that are designated as market order, and *Passive Interaction* take place when the fluid price is defined as limit or stop order where there is no

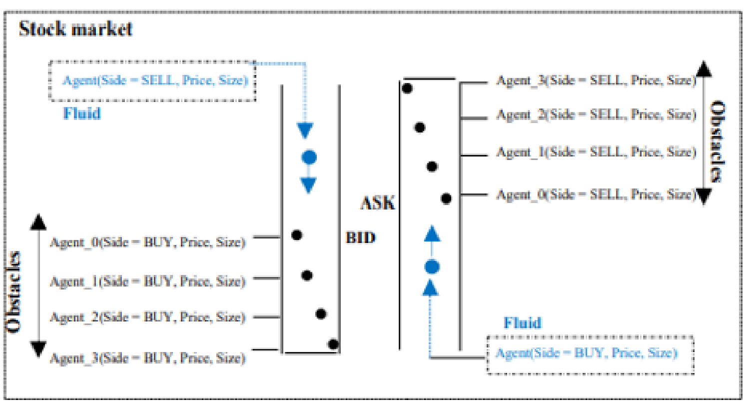

Figure 5. Free body diagram of a fluid interacting with obstacles (46)

collision between the order-book's Bid/Ask (47). This scenario is illustrated using an economic analogue of a free body diagram in figure 5. We can also utilise Stokes' Law to find an economic equivalent of the viscous parameter if we are aware of the stock market's economic physical characteristics and those of every new agent (47). Given that there are no equivalent terms for radius and gravity in economics, those terms could be omitted to simplify the process and the expression for the economic analogue of dynamic viscosity is given below:

$$\mu_{\mathrm{fluid}} = \frac{\mathrm{S_{obstacle}} * \mathrm{P_{obstacle}} - \mathrm{S_{order}} * \mathrm{P_{order}}}{\mathrm{V} * \upsilon} \qquad \text{Eq.18}$$

Where S, P, V and υ correspond to the size, price, volume of transaction and velocity of the agent. If the" buying force" is dominant, $\mathrm{S_{obstacle}} = S_{ask}$ and $\mathrm{P_{obstacle}} = P_{ask}$. If the "selling force is dominant, then $\mathrm{S_{obstacle}} = S_{bid}$ and $\mathrm{P_{obstacle}} = P_{bid}$. And the economic analogue of the Reynolds Number (48, 49):

$$N_R = \frac{\frac{S_{fluid} * P_{fluid}}{S_{obstacle} * P_{obstacle}} . \upsilon^2 . l}{1 - \frac{S_{fluid} * P_{fluid}}{S_{obstacle} * P_{obstacle}}} \qquad \text{Eq.19}$$

Whereas $\mathbb{P} = \frac{S_{fluid} * P_{fluid}}{S_{obstacle} * P_{obstacle}}$ denotes the conditional probability of having a perfect collision, l represents the characteristic length, which is equivalent to the necessary travel length to make the stock market price change (48, 49). To validate this model two simulations were executed and the results linking the Reynolds Number, market speed variation (change in prices for every time t) and $\mathbb{P}$ could be observed in the 3D surface plot in figure 6. The Reynolds number function of market speed and collision probability is shown in Figure 6a. Two local maxima can be evaluated when the likelihood of a collision and the absolute value of market speed are both maximum. The Reynolds number function of market spread and collision probability are shown in Figure 6b. One local maximum can be evaluated when the dispersion widens, and the likelihood of a collision is at its highest. Consequently, through computer simulations, we can observe the same property as that provided by Eq.19, namely that the larger Reynolds number value is a function of the market's speed, the likelihood that supply and demand will collide, and that the larger spread acts as a catalyst for this collision (47).

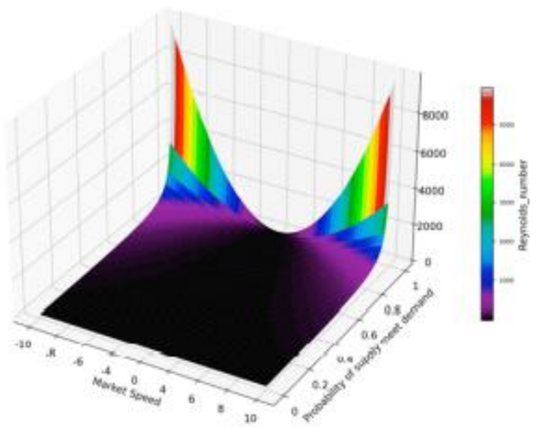

Figure 6a. 3D Surface plot of Reynolds Numbers against market speed variation (change in prices for every time t) and the probability of having a collision $\mathbb{P}_{collision}$ (46).

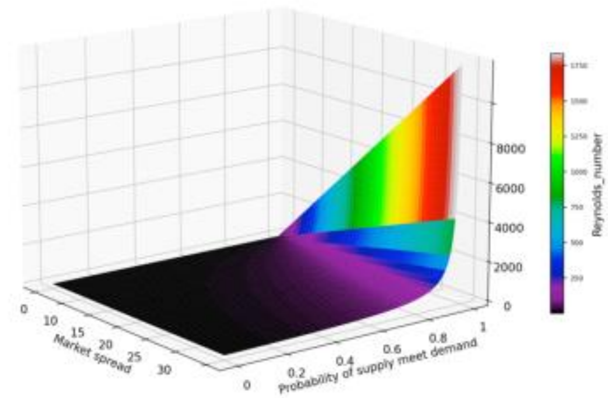

Figure 6b. 3D Surface plot of Reynolds Numbers against market Bid/Ask spread ($l = |\{p_0\} - \{p_1\}|$ *for every time* t*) and the probability of having a collision* $\mathbb{P}_{collision}$ *(46).*

Another literature worth highlighting is *Stock markets: A view from soft matte w*ritten by Antonio M. in 2020 (50), in which the stock price and trading of big and stable companies was modelled using the physics of many-particle systems. Two sets of US and European stocks were used in the study with a pair distribution close to 1, which means there is no direct interaction between stocks, similar to an ideal gas of particles. A few parameters including *mean-squared price displacement* (MSPD); the price correlation function, equivalent to the *intermediate scattering function* (ISF); the price fluctuation distribution; and two parameters for collective motions were used to compare the behaviour of Brownian particles and the financial market (50, 51). Three sets of stocks, including US stocks, UK stocks and European (including UK) stocks were used. The structure and dynamics of the portfolios were measured, and logarithm of price is used for non-dimensionalisation such that stocks in different currencies could be compared. Log-price distributions and the pair distribution function are used to analyse the structure, whereas correlation functions such as the mean-squared log return (log-price difference), *Van Hove functions*, and observables are used for dynamics. In structural terms, the *pair function distribution* of stock $g(\omega)$ is very close to 1 (50). Proving that there is little to no correlation between stocks, which has the same structure as an ideal gas. For dynamics, data form the stock markets are fed into a generalized ISF, i.e., the Fourier transform of MSPD. The ISF of different wave numbers are compared in figure 7 with decaying correlation function, as expected for a fluid state. However, the decay is more stretched than a simple exponential and is found to behave similar to the correlation function in undercooled liquids, governed by the *Khoulrausch stretched exponential* (52). In addition, the absence of velocity correlations and linear growth of MSPD is comparable to the Brownian motion of independent particles. Based on these similarities, Antonio *et al.* proposed two physical system that could explain the stock market qualitatively (50). The first one consists of a colloid with short-range attractions that, when the attraction strength is increased, produces reversible clusters that percolate in the presence of sufficient interactions. As a matter of fact, the MSD of this system increases linearly for strong attractions, where the bonds influence particle diffusion even over very short timescales but also exhibit a high degree of cooperative behaviour. However, in sharp contrast to stock markets, gels exhibit strong structural relationships. The second system that can be compared to stocks is an extension of the perfect gas; it is made up of rigid stars that are infinitely thin and lack structural correlations but can exhibit a sizable slowdown for high densities (50).

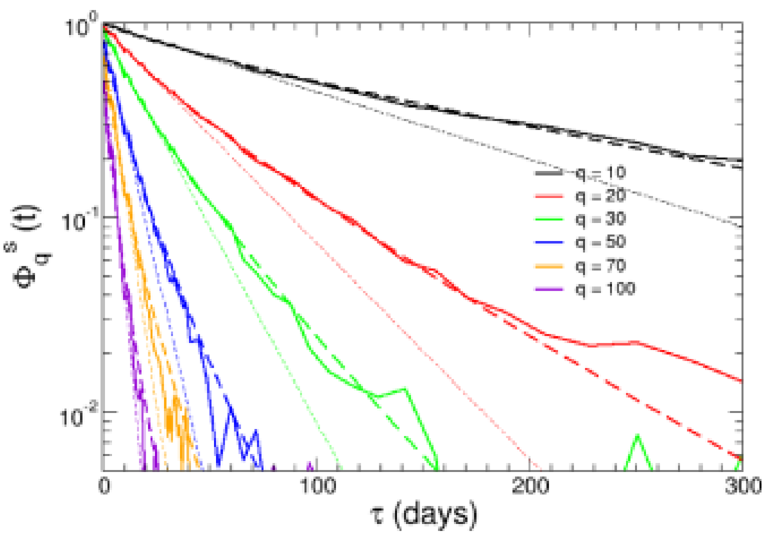

Figure 7. Self part of the intermediate scattering function of the US stocks for different wave numbers, increasing from top to bottom as labeled. The thin lines show the simple exponential fitting of the initial decay, and the thick lines correspond to the fitting of the stretched exponential (49).

In the paper *An econophysics approach to analyse uncertainty in financial markets: an application to the Portuguese stock market,* the stock market is analysed as a physical system in entropy perspective, which is widely used as a measure of dispersion, uncertainty and disorder in engineering physics (53). The variance has always played a crucial part in the understanding of risk and uncertainty. Entropy, on the other hand, can be an alternative measure of dispersion, and Soofi thinks that it is important to be cautious when using variance as a measure of uncertainty (54). Entropy is a gauge of how far away from the uniform distribution the density $p_X(x)$ is. It assesses the "utility" of substituting $p_X(x)$ for the uniform distribution in order to estimate uncertainty (55). The variance represents the average deviation of the probability distribution's results from the mean. Both measures, according to (56), indicate concentration, but their individual metrics of concentration differ. Contrary to the variance, which simply assesses concentration around the mean, entropy assesses diffuseness of the density regardless of where concentration occurs. Similar to the analysis performed by Leton and Gruber (57), Andreia *et al.* rejected the null hypothesis that the rates of return of a diversified portfolio follows a normal distribution (53). High levels of kurtosis and skewness can be observed, which is consistent with what we have discussed in section 3.2. Eq.20 below yield the entropy of a normal distribution $NH(X)$ and it used to compare entropy and traditional standard deviation in uncertainty analysis (53). Since entropy uses a lot more information about the probability distribution than variance, it can be concluded that it is more sensitive to diversification and is a more general uncertainty measure.

$$NH(X) = \int p_X(x) log\sqrt{2\pi}\sigma \, dx + \int p_X(x) \frac{(x-\bar{x})^2}{2\sigma^2} dx = \log(\sqrt{2\pi e}\sigma) \qquad \text{Eq.20}$$

4 Discussions

Several connections between fluid mechanics and financial markets are shown in Section 3. This section will examine the benefits and drawbacks of using these formulations as well as prospective directions for further research.

It is worth noting that the motivation to relate the financial market with fluid mechanics expands far beyond academic curiosity. In the world of finance, any investment that outperforms the broader market is considered a good investment. Whereas a group of top hedge funds generates an average annual return of 15.5% over the past five years (58). Alternative methods of modelling, including the governing law of fluid mechanics, are explored to improve returns and simplify computation processes. Moreover, it could be beneficial to obtain a deeper insight to the physical perspective of financial algorithms and make appropriate adjustments, given that some econometric models are derived from models such as Brownian motion.

Fractality and scaling are some of the key relationships between fluid turbulence and financial markets. To best model the volatile behaviour of the financial market with intermittent extreme events, Mandelbrot attacked the issue by taking ratios of stochastic process. From prior studies it is proven that the multiplicative interaction in time series results in fractality, which act as an obstacle for traditional forecasting model to give accurate results. The inability for econometric models such as ARCH and ARMA to capture fractality and intermittency of the financial market has been addressed in numerous literatures, and solutions derived from fluid turbulence are given. The state transition forecasting algorithm for exchange market proposed by Richards is an excellent example (8), which yield errors smaller than general econometric models in over 60% of the simulations. However, AR1 and GARCH models have also produced decent results and it is understandable to wonder why they are able to forecast as well as they can given that exchange rates are fractals rather than random walks or ARCH processes. The *Kalman filter's* ability to at least partially capture state changes is one factor contributing to this. The fact that they parameterize some of the data's inhomogeneity could perhaps hold the key to the solution. The scaling of the ARCH functions utilised in the forecasting models reveals a non-zero codimension. Similar to this, the Kalman filter's time varying AR1 parameter also exhibits fractal behaviour. These models are non-fractal, but when fitted to the data, they do reflect some of the intermittency. Though this effect is beneficial, these statistical models are still less capable to model catastrophic changes in the financial market. The intermittent nature of the stock market also directly contradicts with the random-walk assumption when deriving the Black-Scholes model, which is part of the reason why there are grounds for profit by manipulating mispriced assets.

One important aspect of fluid mechanics implementation into finance is its computational simplicity and efficiency, but mixed performances could be seen in previous literatures. For instance, in (35) complexity arises when solving the linear Fredholm equation using some forward differences method with N nodes in the space domain and M nodes in the time domain. This led to an order of complexity of $O(M^3)$ for a single value of space S and $O(2kL(M+N))$ for all strikes and maturities, where k is the number of iterations and L is a constant from the numerical method used. Non-linear equations need to be solved for American options and the situation worsens. Other numerical methods such as Crank-Nicolson or backward differences could be adopted but there is a compromise between speed and accuracy.

It is intriguing to observe that the majority of literature regarding the link between fluid mechanics and financial market were published before the 2000s, perhaps due to increasing interest with pure statistical methods alongside advancements in artificial intelligence (AI) technologies. It is apparent that the approaches based on fluid mechanics are primarily connected to the financial statistics that apply more to trading than fundamental investing. Therefore, it is understandable that computationally intensive fluid mechanics approaches would be left undeveloped as it contradicts with the principal of minimising risks and exposure by executing trades in unit of milliseconds. The majority of earlier studies have explored the connection between fluid turbulence and the stock market, but they lack an assessment of how useful these connections are in practise. To gauge interest for further development, for example, their capacity to consistently ensure return and their adaptability to respond to macroeconomic events and make adjustments fall short. By all means, physics and fluid mechanics theories serve as a great toolkit for solving analytical problems with economic models. The economic counterpart of fluid characteristics may offer additional insights that conventional economic models do not take into account, even while it is insufficient to serve as the primary indicator for investment or trading decisions.

5 Conclusions

In this literature review, the laws of the financial market and the underlying mathematical formulations were introduced. Some insights into the Nobel Prize-winning Black-Scholes model and the basic mechanism of the corresponding option pricing were given. Along with some of the most accurate autoregressive speculative models, an overview of the most frequently used and straightforward forecasting model, CAPM, was provided.

Links between the laws of fluid mechanics and financial markets could be demonstrated in three ways. Firstly, the theories of fluid motion played an important role in the development of stochastic processes in econometrics primarily by adopting Geometric Brownian Motion in mathematical models with the premise that asset values have constant volatilities. However, this oversimplified assumption has been the fundamental impediment to more precise forecasting which is why the chaotic nature of the financial market have been attacked in a fluid turbulence perspective, where both scenarios share traits of multifractality and intermittency. Various financial models and forecasting algorithms were derived in previous literatures using the characteristics of fluid behaviours. Most indicate decent theoretical possibilities but restricted implementations, with the top-performing model being the state-transition based FX algorithm which outperforms or makes even with conventional statical methods in two-third of the backtested data.

Finally, a particle system viewpoint was covered, with a focus on particle collision and entropy. Entropy is a more comprehensive measure of uncertainty than variance because it makes use of a lot more information about the probability distribution, making it more sensitive to diversification. Economic analogue of Reynolds number and viscosity were also explained, where they serve as alternative measures to dispersion. The discussion compared the merits and limitations on fluid mechanics applications to the financial market and highlighted possible reasons behind the slowdown of relevant studies.

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